TWMS J. Pure Appl. Math. V.16, N.1, 2025, pp.137-148
DOI: 10.30546/2219-1259.16.1.2025.137

CONVERGENCE, STABILITY AND DATA DEPENDENCE FOR FIXED POINTS OF GENERALIZED NONEXPANSIVE MAPPINGS WITH APPLICATIONS

C. WANG[1]

Abstract. In this paper, we construct the convergence and stability of iterative algorithms for fixed points under a weaker condition of the generalized nonexpansive mappings in Banach spaces. Some new data dependence theorems are also presented. Finally, our results are applied to consider the existence, uniqueness and approximation of solutions for a class of nonlinear fractional differential equations.

Keywords: convergence, stability, fixed points, generalized nonexpansive mappings, iterative algorithms, nonlinear fractional differential equations.

AMS Subject Classification: 47H09, 47H10, 54H25.

1. Introduction

Let K be a nonempty, closed and convex subset of a Banach space X, and $T : K \to K$ is a self mapping. $Fix(T) = \{x \in K : Tx = x\}$ is the fixed point set of T. The mapping T is called contractive if there exists a real number $k \in [0, 1)$ such that

$$\|Tx - Ty\| \leq k\|x - y\| \tag{1}$$

for all $x, y \in K$. If $k = 1$ in (1), then T is said to be nonexpansive. The mapping T is called generalized nonexpansive if there exist five real numbers $a, b, c, d, e \in [0, 1]$ such that

$$\|Tx - Ty\| \leq a\|x - y\| + b\|x - Tx\| + c\|y - Ty\| + d\|x - Ty\| + e\|y - Tx\| \tag{2}$$

for all $x, y \in K$.

Remark 1.1. *(i) Let $b = c = d = e = 0$ and $a < 1$ in (2), then T is a contractive mapping.*
(ii) Let $b = c = d = e = 0$ and $a = 1$ in (2), then T is a nonexpansive mapping.

The Banach contraction theorem tells us that any contractive mapping has a unique fixed point and the Picard iterative algorithm [17] converges to the fixed point. However, Picard iterative algorithm often fails to converge to fixed points of nonexpansive mappings. So, many authors constructed various iterative algorithms to weakly or strongly converge to fixed points of nonexpansive mappings, for example, the Mann algorithm [15], the Ishikawa algorithm [9],

[1]School of Mathematics and Statistics, Nanjing University of Information Science and Technology, China
e-mail: wangchaosx@nuist.edu.cn
Manuscript received July 2024.

the Noor algorithm [23] and etc. Independent of these algorithms, in [1] introduced a two-step iterative algorithm:

$$\begin{cases} x_0 \in K, \\ x_{n+1} = (1-\alpha_n)Tx_n + \alpha_n Ty_n, \\ y_n = (1-\beta_n)x_n + \beta_n Tx_n \end{cases} \tag{3}$$

for all $n \geq 0$, where $\{\alpha_n\}$, $\{\beta_n\} \subset (0, 1,)$ the authors proved that the iterative algorithm (3) converges faster than many other algorithms to fixed points of nonexpansive mappings. Furthermore, in [19] designed a three-step iterative algorithm:

$$\begin{cases} x_0 \in K, \\ x_{n+1} = (1-\alpha_n)Tx_n + \alpha_n Ty_n, \\ y_n = (1-\beta_n)z_n + \beta_n Tz_n, \\ z_n = (1-\gamma_n)x_n + \gamma_n Tx_n \end{cases} \tag{4}$$

for all $n \geq 0$, where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\} \subset (0, 1)$. Under some conditions on coefficients $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$, they showed that the iterative algorithm (4) converges better than the iterative algorithm (3) to fixed points of contractive mappings.

In 1973, in [8] proved that if

$$a + b + c + d + e < 1, \tag{5}$$

then the Picard iterative algorithm converges to a unique fixed point of generalized nonexpansive mappings in complete metric spaces. And then, in [21, 22] used the Krasnoselskii iterative algorithm [11] to approximate the common fixed point of a pair of generalized nonexpansive mappings in complete convex metric spaces and uniformly convex metric spaces. In [12] investigated the convergence of the iterative algorithm (4) for generalized nonexpansive mappings with

$$a + 2b + c + 2d + 3e < 1 \tag{6}$$

and some conditions on $\{\alpha_n\}$ and $\{\beta_n\}$. Inspired by [6, 12], in [14] studied the stability and data dependence of the iterative algorithm (4) for contractive mappings in Banach spaces with some conditions on $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$. Very recently, without any condition on $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$, in [7] reproduced the results in [12, 14], they gave some convergence theorems of the iterative algorithm (4) for contractive mappings and the iterative algorithm (3) for generalized nonexpansive mappings with the condition (6), respectively. Meanwhile they discussed the stability of the iterative algorithm (4) for contractive mappings, data dependence results of the iterative algorithm (4) for contractive mappings, and the iterative algorithm (3) for generalized nonexpansive mappings with

$$a + 2b + 2c + 3d + 3e < 1 \tag{7}$$

were also established.

The purpose of this paper is to improve many known results as the following four aspects:

(i) In Section 2, we consider convergence theorems of the iterative algorithms (3) and (4) for generalized nonexpansive mappings with the weaker condition (5) (it is obviously that the generalized nonexpansive mapping given in Example 3.1 satisfies the condition (5), but does not satisfy the condition (6) or (7)). Based on convergence theorems in Section 2, we discuss the stability of the iterative algorithms (3) and (4) for generalized nonexpansive mappings without any condition on $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$.

(ii) In Section 3, we study data dependence results of the iterative algorithms (3) and (4) for generalized nonexpansive mappings, which give some much better upper bound estimates.

(iii) In Section 4, based on the theorems in Section 2 and Section 3, we discuss the existence and uniqueness of solutions for a class of nonlinear fractional differential equations. Meanwhile, we use the iterative algorithm (4) to approximate a unique solution.

Next, we recall some concepts and results.

Definition 1.1. [5] *Let $T : X \to X$ be a self mapping. For arbitrary $x_0 \in X$, the sequence $\{x_n\}$ is produced by*

$$x_{n+1} = f(T, x_n) \tag{8}$$

for all $n \geq 0$. Assume that $\{x_n\}$ converges to a fixed point p of T. For any sequence $\{y_n\} \subset X$ and set

$$\epsilon_n = d(y_{n+1}, f(T, y_n))$$

for all $n \geq 0$. We say that the iterative algorithm (8) is $T-$stable (or stable with respect to T) if and only if

$$\lim_{n\to\infty} \epsilon_n = 0 \quad \text{if and only if} \quad \lim_{n\to\infty} y_n = p.$$

Definition 1.2. [5] *Let $T, \tilde{T} : K \to K$ be two self mappings. Then $\tilde{T}$ is called an approximate operator of T if there exists an $\epsilon > 0$ such that*

$$\|Tx - \tilde{T}x\| \leq \epsilon.$$

Lemma 1.1. [5] *Suppose $\{a_n\}$ and $\{b_n\}$ are two nonnegative real sequences satisfying*

$$a_{n+1} \leq \delta a_n + b_n, n \geq 0,$$

where $\delta \in [0, 1]$ and $\lim_{n\to\infty} b_n = 0$. Then $\lim_{n\to\infty} a_n = 0$.

2. Convergence and stablility results

In this section, we will establish the convergence and stability of the iterative algorithms (3) and (4) for generalized nonexpansive mappings.

Theorem 2.1. *Suppose $T : K \to K$ is a generalized nonexpansive mapping with the condition (5). Then the sequence $\{x_n\}$ defined by the iterative algorithm (4) converges to the unique fixed point p of T.*

Proof. From Theorem 1 in [8], we know that T has a unique fixed point p and for any $y, x \in K$,

$$\|Ty - Tx\| \leq a\|y - x\| + b\|y - Ty\| + c\|x - Tx\| + d\|y - Tx\| + e\|x - Ty\|.$$

Combining formula (2), we know that

$$\|Tx - Ty\| \leq a\|x - y\| + \frac{b+c}{2}\left[\|x - Tx\| + \|y - Ty\|\right] + \frac{d+e}{2}\left[\|x - Ty\| + \|y - Tx\|\right].$$

Therefore,

$$\begin{aligned}\|Tx - p\| &= \|Tx - Tp\| \\ &\leq a\|x - p\| + \frac{b+c}{2}\left[\|x - Tx\| + \|p - Tp\|\right] + \frac{d+e}{2}\left[\|x - Tp\| + \|p - Tx\|\right] \\ &\leq a\|x - p\| + \frac{b+c}{2}\left[\|x - p\| + \|p - Tx\|\right] + \frac{d+e}{2}\left[\|x - p\| + \|p - Tx\|\right],\end{aligned}$$

which implies

$$\|Tx - p\| \leq \frac{a + \frac{b+c}{2} + \frac{d+e}{2}}{1 - \frac{b+c}{2} - \frac{d+e}{2}} \|x - p\| = \theta \|x - p\|,$$

where $\theta = \frac{2a+b+c+d+e}{2-b-c-d-e}$. It follows from the condition (5) that $\theta \in [0, 1)$. By the iterative algorithm (4), we have

$$\|z_n - p\| = \|(1-\gamma_n)x_n + \gamma_n Tx_n - p\| \leq (1-\gamma_n)\|x_n - p\| + \gamma_n \theta \|x_n - p\| \leq [1 - \gamma_n(1-\theta)] \|x_n - p\| \tag{9}$$

and

$$\|y_n - p\| = \|(1-\beta_n)z_n + \beta_n Tz_n - p\| \leq [1 - \beta_n(1-\theta)] \|z_n - p\|. \tag{10}$$

Inserting (9) into (10), we obtain

$$\|y_n - p\| \leq [1 - \beta_n(1-\theta)] \cdot [1 - \gamma_n(1-\theta)] \|x_n - p\|. \tag{11}$$

By (4) and (11), we get

$$\begin{aligned}
\|x_{n+1} - p\| &= \|(1-\alpha_n)Tx_n + \alpha_n Ty_n - p\| \\
&\leq (1-\alpha_n)\theta\|x_n - p\| + \alpha_n \theta \|y_n - p\| \\
&\leq \theta \left\{1 - \alpha_n + \alpha_n [1 - \beta_n(1-\theta)] \cdot [1 - \gamma_n(1-\theta)]\right\} \|x_n - p\| \\
&\leq \theta \left[1 - \alpha_n\gamma_n(1-\theta) - \alpha_n\beta_n(1-\theta) + \alpha_n\beta_n\gamma_n(1-\theta)^2\right] \|x_n - p\| \qquad (12) \\
&\leq \theta \left[1 - \alpha_n\beta_n\gamma_n(1-\theta)^2\right] \|x_n - p\|. \qquad (13)
\end{aligned}$$

Notice that $1 - \alpha_n\beta_n\gamma_n(1-\theta)^2 < 1$. Then

$$\|x_{n+1} - p\| \leq \theta \|x_n - p\|,$$

which implies

$$\|x_{n+1} - p\| \leq \theta^{n+1} \|x_0 - p\|. \tag{14}$$

Taking limit on both sides of (14), we have

$$\lim_{n\to\infty} \|x_{n+1} - p\| = 0, \tag{15}$$

i.e. $\{x_n\}$ converges to the unique fixed point p of T. □

Remark 2.1. *Theorem 2.1 extends some results in [8, 10-12, 21, 22], and Remark 1.1. in [7] and Theorem 4.5. in [14] to the case of generalized nonexpansive mappings.*

Corollary 2.1. *Suppose $T : K \to K$ is a generalized nonexpansive mapping with the condition (5). Then the sequence $\{x_n\}$ defined by the iterative algorithm (3) converges to the unique fixed point p of T.*

Proof. Let $\gamma_n = 0$ in (12). Then we also have

$$\|x_{n+1} - p\| \leq \theta \|x_n - p\|.$$

Similar to the proof of Theorem 2.1, we know that $\{x_n\}$ converges to the unique fixed point p of T. □

Remark 2.2. *Since we replace the condition "$a+2b+c+2d+3e < 1$" by "$a+b+c+d+e < 1$", Corollary 2.5 improves Remark 2 in [7] and Theorem 3.1 in [12].*

Example 2.1. Suppose K, X and T are defined as Example 2 in [7]. From [7], we know that T is a generalized nonexpansive mapping with

$$a = \frac{\beta}{\alpha(1-2\gamma)}, \quad b = e = 0, \quad c = d = \frac{\gamma}{1-2\gamma}.$$

By Theorem 2.1 and Corollary 2.1, if

$$a + b + c + d + e = \frac{\beta}{\alpha(1-2\gamma)} + \frac{2\gamma}{1-2\gamma} < 1,$$

i.e.

$$\gamma < \frac{1}{4}\left(1 - \frac{\beta}{\alpha}\right),$$

then the iterative algorithms (3) and (4) both converge to the fixed point $x_*(t)$ of T. In this case, for fixed α and β, we can choose more real numbers γ and ρ than [7]. Indeed, γ must satisfy $\gamma < \frac{1}{5}\left(1 - \frac{\beta}{\alpha}\right)$ in [7].

Now, we will show that the iterative algorithms (3) and (4) are both stable with respect to generalized noneexpansive mappings with the condition (5).

Theorem 2.2. *Suppose $T : K \to K$ is a generalized nonexpansive mapping with the condition (5). Then the sequence $\{x_n\}$ defined by the iterative algorithm (4) is $T-$ stable.*

Proof. From Theorem 2.1, $\{x_n\}$ converges to the unique fixed point p. Assume that $\{u_n\}$ is an arbitrary sequence in K. Define

$$\begin{cases} \epsilon_n = \|u_{n+1} - (1-\alpha_n)Tu_n - \alpha_n Tv_n\|, \\ v_n = (1-\beta_n)w_n + \beta_n Tw_n, \\ w_n = (1-\gamma_n)u_n + \gamma_n Tu_n. \end{cases} \tag{16}$$

(i) Let $\lim_{n\to\infty} \epsilon_n = 0$. Then we get

$$\|u_{n+1} - p\| \le \|(1-\alpha_n)Tu_n + \alpha_n Tv_n - p\| + \epsilon_n. \tag{17}$$

Similar to the proof of (13), from (16), we have

$$\begin{aligned} \|(1-\alpha_n)Tu_n + \alpha_n Tv_n - p\| &\le \theta\left[1 - \alpha_n\beta_n\gamma_n(1-\theta)^2\right]\|u_n - p\| \\ &\le \theta\|u_n - p\| \end{aligned} \tag{18}$$

By (17) and (18), we obtain

$$\|u_{n+1} - p\| \le \theta\|u_n - p\| + \epsilon_n.$$

It follows from Lemma 1.1 that $\lim_{n\to\infty} \|u_{n+1} - p\| = 0$, i.e. $\lim_{n\to\infty} u_n = p$.

(ii) Conversely, assume that $\lim_{n\to\infty} u_n = p$. Then

$$\begin{aligned} 0 \le \epsilon_n &= \|u_{n+1} - (1-\alpha_n)Tu_n - \alpha_n Tv_n\| \\ &\le \|u_{n+1} - p\| + \theta\|u_n - p\|, \end{aligned}$$

which implies $\lim_{n\to\infty} \epsilon_n = 0$.

From (i), (ii), and Definition 1.1, we know that $\{x_n\}$ is $T-$ stable. □

Remark 2.3. *Theorem 2.2 extends Theorem 4 in [7] and Theorem 4.6 in [14] to the case of generalized nonexpansive mappings.*

Corollary 2.2. *Suppose $T : K \to K$ is a generalized nonexpansive mapping with the condition (5). Then the sequence $\{x_n\}$ defined by the iterative algorithm (3) is $T-$ stable.*

Example 2.2. Let $K = [0,1] \subset X = \mathbb{R}$, define a mapping $T : K \to K$ by

$$Tx = \begin{cases} \frac{1}{10}x, & x \in [0, 1/2], \\ \frac{1}{12}x, & x \in (1/2, 1], \end{cases}$$

It is obvious that T is a generalized nonexpansive mapping with

$$a = 0, \quad b = c = \frac{2}{7}, \quad d = e = \frac{1}{7}.$$

So, $a + b + c + d + e = \frac{6}{7} < 1$. By Theorem 2.1, we know that the iterative algorithms (3) and (4) both converge to the fixed point $p = 0$. From Theorem 2.5 and Corollary 2.2, the iterative algorithms (3) and (4) are both $T-$ stable.

In order to reflect the stability of the iterative algorithms (3) and (4) from the numerical point of view, we set

$$\alpha_n = \gamma_n = \sqrt{\frac{n+1}{n+2}}, \quad \beta_n = \frac{1}{n+3}, \quad u_n = \frac{1}{(n+2)^2}.$$

Obviously, we have $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0,1)$ and $\{u_n\} \subset K$. Iteration results of $\{\epsilon_n^i\}$ $(i = 1, 2)$ are shown in Fig.1, where $\{\epsilon_n^1\}$ is given by the iterative algorithm (3) and $\{\epsilon_n^2\}$ is given by the iterative algorithm (4).

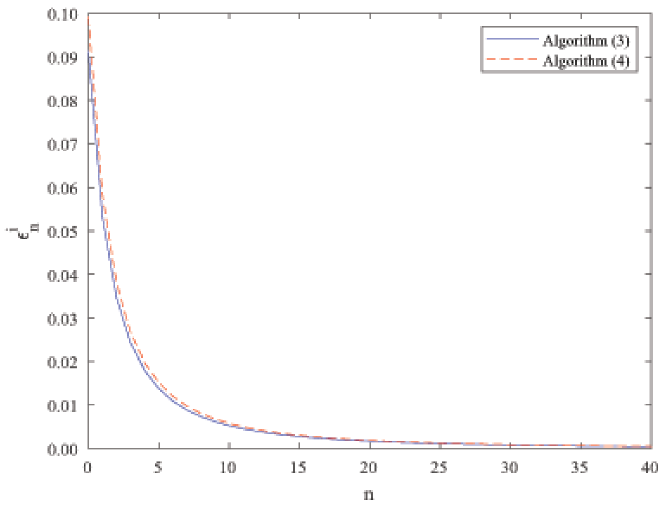


Figure 1. Iteration results of $\{\epsilon_n^i\}$ with $\{u_n\}$.

Figure 1 shows that $\lim_{n\to\infty} \epsilon_n^i = 0$ for $i = 1, 2$. Therefore, the iterative algorithms (3) and (4) are both numerically stable.

3. Date dependence results

In this section, we will study the dependence of the iterative algorithms (3) and (4) for generalized nonexpansive mappings.

Theorem 3.1. *Suppose T is defined as Theorem 2.1. Let $\tilde{T}$ be an approximate mapping of T with error $\epsilon > 0$. Let $\{x_n\}$ be an iterative sequence defined by the iterative algorithm (4) and* *construct an iterative sequence* $\{\tilde{x}_n\}$ *as follows*

$$\begin{cases} \tilde{x}_0 \in K, \\ \tilde{x}_{n+1} = (1-\alpha_n)\tilde{T}\tilde{x}_n + \alpha_n \tilde{T}\tilde{y}_n, \\ \tilde{y}_n = (1-\beta_n)\tilde{z}_n + \beta_n \tilde{T}\tilde{z}_n, \\ \tilde{z}_n = (1-\gamma_n)\tilde{x}_n + \gamma_n \tilde{T}\tilde{x}_n \end{cases} \tag{19}$$

for all $n \geq 1$. *If* $Tp = p$ *and* $\tilde{T}\tilde{p} = \tilde{p}$ *such that* $\lim_{n\to\infty} \tilde{x}_n = \tilde{p}$, *then*

$$\|p - \tilde{p}\| \leq \left(\frac{1+2\theta}{1-\theta}\right)\epsilon,$$

where $\theta = \frac{2a+b+c+d+e}{2-b-c-d-e} \in [0,1)$.

Proof. From Theorem 2.1, $\lim_{n\to\infty} x_n = p$. By (4), (11) and (19), we obtain

$$\begin{aligned} \|x_{n+1} - \tilde{x}_{n+1}\| &= \|(1-\alpha_n)Tx_n + \alpha_n Ty_n - (1-\alpha_n)\tilde{T}\tilde{x}_n - \alpha_n \tilde{T}\tilde{y}_n\| \\ &\leq (1-\alpha_n)\|Tx_n - \tilde{T}\tilde{x}_n\| + \alpha_n\|Ty_n - \tilde{T}\tilde{y}_n\| \\ &\leq (1-\alpha_n)\|Tx_n - T\tilde{x}_n\| + (1-\alpha_n)\epsilon + \alpha_n\|Ty_n - T\tilde{y}_n\| + \alpha_n\epsilon \\ &= (1-\alpha_n)\|Tx_n - T\tilde{x}_n\| + \alpha_n\|Ty_n - T\tilde{y}_n\| + \epsilon, \end{aligned} \tag{20}$$

$$\begin{aligned} (1-\alpha_n)\|Tx_n - T\tilde{x}_n\| &\leq (1-\alpha_n)\|Tx_n - p\| + (1-\alpha_n)\|T\tilde{x}_n - p\| \\ &\leq (1-\alpha_n)\theta\|x_n - p\| + (1-\alpha_n)\theta\|\tilde{x}_n - p\| \\ &\leq (1-\alpha_n)\theta\|x_n - p\| + (1-\alpha_n)\theta\|\tilde{x}_n - \tilde{p}\| + (1-\alpha_n)\theta\|p - \tilde{p}\|, \end{aligned} \tag{21}$$

and

$$\begin{aligned} \alpha_n\|Ty_n - T\tilde{y}_n\| &\leq \alpha_n\|Ty_n - p\| + \alpha_n\|T\tilde{y}_n - p\| \\ &\leq \alpha_n\theta\|y_n - p\| + \alpha_n\theta\|\tilde{y}_n - p\| \\ &\leq \alpha_n\theta\left[1-\beta_n(1-\theta)\right]\cdot\left[1-\gamma_n(1-\theta)\right]\|y_n - p\| + \alpha_n\theta\cdot\|\tilde{y}_n - p\| \end{aligned} \tag{22}$$

Notice that

$$\begin{aligned} \|\tilde{y}_n - p\| &= \|(1-\beta_n)\tilde{z}_n + \beta_n\tilde{T}\tilde{z}_n - p\| \\ &\leq (1-\beta_n)\|\tilde{z}_n - p\| + \beta_n\|\tilde{T}\tilde{z}_n - T\tilde{z}_n + T\tilde{z}_n - p\| \\ &\leq (1-\beta_n+\beta_n\theta)\|\tilde{z}_n - p\| + \beta_n\epsilon \\ &\leq (1-\beta_n+\beta_n\theta)[(1-\gamma_n+\gamma_n\theta)\|\tilde{x}_n - \tilde{p}\| + (1-\gamma_n+\gamma_n\theta)\|p-\tilde{p}\| \\ &\quad + \gamma_n\epsilon] + \beta_n\epsilon \\ &\leq (1-\beta_n+\beta_n\theta)\cdot(1-\gamma_n+\gamma_n\theta)\|\tilde{x}_n - \tilde{p}\| + (1-\beta_n+\beta_n\theta) \\ &\quad \times (1-\gamma_n+\gamma_n\theta)\cdot\|p-\tilde{p}\| + [\beta_n + \gamma_n(1-\beta_n+\beta_n\theta)]\cdot\epsilon. \end{aligned} \tag{23}$$

Then, by (20)-(23), we have

$$\begin{aligned} \|x_{n+1} - \tilde{x}_{n+1}\| &\leq [\theta - \alpha_n\theta + \alpha_n\theta(1-\beta_n+\beta_n\theta)\cdot(1-\gamma_n+\gamma_n\theta)]\,\|x_n - p\| \\ &\quad + [\theta - \alpha_n\theta + \alpha_n\theta(1-\beta_n+\beta_n\theta)\cdot(1-\gamma_n+\gamma_n\theta)]\,\|\tilde{x}_n - \tilde{p}\| \\ &\quad + [\theta - \alpha_n\theta + \alpha_n\theta(1-\beta_n+\beta_n\theta)\cdot(1-\gamma_n+\gamma_n\theta)]\,\|p - \tilde{p}\| \\ &\quad + [1 + \alpha_n\beta_n\theta + \alpha_n\gamma_n(1-\beta_n+\beta_n\theta)\theta]\,\epsilon \qquad (24) \\ &\leq \theta\|x_n - p\| + \theta\|\tilde{x}_n - \tilde{p}\| + \theta\|p - \tilde{p}\| + (1+2\theta)\epsilon. \qquad (25) \end{aligned}$$

Taking limit on both sides of (25), we get

$$\|p - \tilde{p}\| \leq \theta \|p - \tilde{p}\| + (1 + 2\theta)\epsilon,$$

which implies

$$\|p - \tilde{p}\| \leq \left(\frac{1 + 2\theta}{1 - \theta}\right) \epsilon.$$

□

Remark 3.1. *(i) Theorem 3.1 extends Theorem 6 in [7] and Theorem 4.7 in [14] to the case of generalized nonexpansive mappings.*

(ii) From (24), if $\lim_{n\to\infty} \alpha_n = 0$*, then we can get*

$$\|p - \tilde{p}\| \leq \left(\frac{1}{1 - \theta}\right) \epsilon.$$

Corollary 3.1. *Suppose T is defined as Theorem 2.1 and $\tilde{T}$ is an approximate mapping of T with error $\epsilon > 0$. Let $\{x_n\}$ and $\{\tilde{x}_n\}$ be two iterative sequences constructed by the iterative algorithms (3) and (19) with $\gamma_n = 0$. If $Tp = p$ and $\tilde{T}\tilde{p} = \tilde{p}$ such that $\lim_{n\to\infty} \tilde{x}_n = \tilde{p}$, then*

$$\|p - \tilde{p}\| \leq \left(\frac{1 + \theta}{1 - \theta}\right) \epsilon,$$

where $\theta = \frac{2a+b+c+d+e}{2-b-c-d-e} \in [0, 1)$.

Proof. Similar to the proof of Theorem 3.1, let $\gamma_n = 0$ in (24), we have

$$\|x_{n+1} - \tilde{x}_{n+1}\| \leq \theta\|x_n - p\| + \theta\|\tilde{x}_n - \tilde{p}\| + \theta\|p - \tilde{p}\| + (1 + \theta)\epsilon.$$

Taking limit on both sides of the above inequality, we obtain

$$\|p - \tilde{p}\| \leq \left(\frac{1 + \theta}{1 - \theta}\right) \epsilon.$$

□

Remark 3.2. *Since we replace the condition "$a+2b+2c+3d+3e < 1$" by "$a+b+c+d+e < 1$", Corollary 3.1 improves Theorem 8 in [7] and Theorem 4.1 in [12]. And Corollary 3.1 gives a much better estimate of upper bound for $\|p - \tilde{p}\|$.*

Example 3.1. Suppose T and $\tilde{T}$ are defined as Example 4.2 in [12] with

$$a = \frac{1}{50}, \quad b = d = \frac{2}{50}, \quad c = 0, \quad d = \frac{3}{50}.$$

In this case, $a + b + c + d + e = 0.16 < 1$. In [12], it gets the following estimate

$$\|x^* - \tilde{x}^*\| \leq 0.75596, \tag{26}$$

where x^* is the fixed point of T, $\tilde{x}^*$ is the fixed point of $\tilde{T}$ and

$$\|x^* - \tilde{x}^*\| = 0.288347, \qquad \epsilon = 0.314986.$$

By Corollary 3.1, we obtain another estimate

$$\|x^* - \tilde{x}^*\| \leq \frac{1 + \frac{2a+b+c+d+e}{2-b-c-d-e}}{1 - \frac{2a+b+c+d+e}{2-b-c-d-e}} \cdot \epsilon \approx 0.38248. \tag{27}$$

which is better than the estimate (26).

Example 3.2. Suppose T and $\tilde{T}$ are defined as Example 5 in [7] with

$$a = \frac{3}{5}, \quad b = e = 0, \quad c = d = 0.05.$$

In this case, $a + b + c + d + e = 0.7 < 1$. In [7], it gets the following estimate

$$\|x_*(t) - \tilde{x}_*(t)\|_\infty \leq 0.00386, \tag{28}$$

where $x_*(t)$ is the fixed point of $T(x(t))$, $\tilde{x}_*(t)$ is the fixed point of $\tilde{T}(x(t))$ and

$$\|x_*(t) - \tilde{x}_*(t)\|_\infty = 0.0009658, \qquad \epsilon = 0.0005.$$

By Corollary 3.1, we obtain another estimate

$$\|x_*(t) - \tilde{x}_*(t)\|_\infty \leq \frac{1 + \frac{2a+b+c+d+e}{2-b-c-d-e}}{1 - \frac{2a+b+c+d+e}{2-b-c-d-e}} \cdot \epsilon \approx 0.00267. \tag{29}$$

which is more effective than the estimate (28).

4. Applications

Recently, many authors [2-4, 6, 13, 16, 18, 20] have used fixed point theorems and iterative algorithms for nonlinear mappings in different spaces to study the existence, approximation, and stability of solutions for various fractional differential equations and integral equations. In this section, by using Theorem 2.1 and Theorem 3.1, we consider the solution of the following nonlinear fractional differential equation [4]:

$$D^\alpha x(t) + D^\beta x(t) = f(t, x(t)), \qquad t \in [0, 1], \quad 0 < \beta < \alpha < 1 \tag{30}$$

with boundary value condition $x(0) = x(1) = 1$, where $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, and D^ρ denotes the Caputo fractional derivative of order ρ, which is defined by

$$D^\rho h(t) = \frac{1}{\Gamma(n - \rho)} \int_0^t (t - s)^{n-\rho-1} h^{(n)}(s) ds$$

where $h : [0, +\infty) \to \mathbb{R}$ is a continuous function and $n - 1 < \rho < n, n = [\rho] + 1$. The Green function of (30) is given by

$$G(t) = t^{\alpha-1} E_{\alpha-\beta}(-t^{\alpha-\beta}).$$

Let $X = C([0, 1], \mathbb{R})$ be a set of real continuous functions defined on $[0, 1]$ with the Bielecki norm

$$\|u - v\| = max_{t\in[0,1]}\{|u(t) - v(t)|e^{-\lambda t}\} \tag{31}$$

where $u, v \in X$ and $\lambda > 0$ is a constant. In this case, $(X, \|\cdot\|)$ is a Banach space. We know that $x(t) \in X$ is a solution of (30) is equivalent to $x(t) \in X$ is a solution of the integral equation

$$x(t) = \int_0^t G(t - s) f(s, x(s)) \, ds \tag{32}$$

for any $t \in [0, 1]$. By denoting

$$(Tx)(t) = \int_0^t G(t - s) f(s, x(s)) \, ds \tag{33}$$

for any $t \in [0, 1]$, we can write (32) as the fixed point form $Tx = x$.

Theorem 4.1. *Suppose there exist nonnegative real constants A, B, C, D and E such that*

$$|f(t,\varphi_1) - f(t,\varphi_2)| \le A|\varphi_1 - \varphi_2| + B|\varphi_1 - T\varphi_1| + C|\varphi_2 - T\varphi_2| + D|\varphi_1 - T\varphi_2| + E|\varphi_2 - T\varphi_1| \tag{34}$$

for any $t \in [0,1]$ and $\varphi_1, \varphi_2 \in \mathbb{R}$, where

$$A + B + C + D + E < \lambda\alpha. \tag{35}$$

Then

(i) the fractional differential equation (30) has a unique solution in X.

(ii) let $x_0(t) \in X$, the iterative algorithm (4) converges uniformly to the unique solution and (4) is T-stable.

Proof. By (33) and (34), for any $\varphi_1, \varphi_2 \in X$, we get

$$\begin{aligned}
|T\varphi_1 - T\varphi_2| &\le \int_0^t G(t-s)\cdot |f(s,\varphi_1(s)) - f(s,\varphi_2(s))|\, ds \\
&\le \int_0^t G(t-s)\cdot [A|\varphi_1-\varphi_2| + B|\varphi_1 - T\varphi_1| + C|\varphi_2 - T\varphi_2| \\
&\le \frac{1}{\alpha}\cdot [A\|\varphi_1-\varphi_2\| + B\|\varphi_1 - T\varphi_1\| + C\|\varphi_2 - T\varphi_2\| \\
&\quad + D|\varphi_1 - T\varphi_2| + E|\varphi_2 - T\varphi_1|] \\
&\le \int_0^t G(t-s)\cdot [A|\varphi_1-\varphi_2|e^{-\lambda s} + B|\varphi_1 - T\varphi_1|e^{-\lambda s} + C|\varphi_2 - T\varphi_2|e^{-\lambda s} \\
&\quad + D|\varphi_1 - T\varphi_2|e^{-\lambda s} + E|\varphi_2 - T\varphi_1|e^{-\lambda s}]\cdot e^{\lambda s}
\end{aligned} \tag{36}$$

Note that $\sup_{t\in[0,1]} \int_0^t G(t-s)ds \le \frac{1}{\alpha}$ ([4]). By (31) and (36), we have

$$\begin{aligned}
|T\varphi_1 - T\varphi_2| &\le \frac{1}{\alpha}\cdot [A\|\varphi_1-\varphi_2\| + B\|\varphi_1 - T\varphi_1\| + C\|\varphi_2 - T\varphi_2\| + D\|\varphi_1 - T\varphi_2\| \\
&\quad + E\|\varphi_2 - T\varphi_1\|]\cdot \int_0^t e^{\lambda s}\, ds \\
&\le \frac{1}{\alpha}\cdot [A\|\varphi_1-\varphi_2\| + B\|\varphi_1 - T\varphi_1\| + C\|\varphi_2 - T\varphi_2\| + D\|\varphi_1 - T\varphi_2\| \\
&\quad + E\|\varphi_2 - T\varphi_1\|]\cdot \frac{e^{\lambda t}}{\lambda}
\end{aligned}$$

which implies

$$\begin{aligned}
\|T\varphi_1 - T\varphi_2\| \le \frac{1}{\lambda\alpha}\cdot [&A\|\varphi_1-\varphi_2\| + B\|\varphi_1 - T\varphi_1\| + C\|\varphi_2 - T\varphi_2\| \\
&+ D\|\varphi_1 - T\varphi_2\| + E\|\varphi_2 - T\varphi_1\|].
\end{aligned}$$

From the condition (35), we know that the mapping $T : X \to X$ is a generalized nonexpansive mapping with the condition (5). It follows from Theorem 2.1 that T has a unique fixed point x^*, i.e., the fractional differential equation (30) has a unique solution $x^* \in X$. According to Theorem 2.1 and Theorem 2.2, it can be inferred that the iterative algorithm (4) converges uniformly to the solution and the algorithm is T-stable. $\square$

Remark 4.1. *Based on the values of parameters A, B, C, D and E, we can choose an appropriate real number λ such that*

$$\lambda > \frac{A+B+C+D+E}{\alpha}.$$

Corollary 4.1. *Suppose f is an L-Lipschitz function with respect to the second variable, i.e.,*

$$|f(t,\varphi_1) - f(t,\varphi_2)| \leq L|\varphi_1 - \varphi_2|$$

for any $t \in [0,1]$ and $\varphi_1, \varphi_2 \in \mathbb{R}$, where $0 < L < \lambda\alpha$. Then the iterative algorithm (4) converges uniformly to a unique solution of the fractional differential equation (30) and (4) is T-stable.

5. Conclusion

The purpose of this paper is to discuss the convergence, stability and data dependence of generalized nonexpansive mappings with the condition (5). We mainly use the iterative algorithm (4) to approximate fixed points for the mappings, and established some convergence, stability and data dependence theorems, which are more general than other previous results. Furthermore, we have presented some numerical examples to illustrate our conclusions. Moreover, using our fixed point theorems, we study the existence and uniqueness of solutions for a class of nonlinear fractional differential equations and provide an effective iterative algorithm to approach a unique solution.

6. Acknowledgement

This work was supported by the Natural Science Foundation of China (No. 11126290). The author thanks the editor and the referees for constructive and pertinent suggestions. The author declares no competing interest.

References

[1] Agarwal, R.P., Regan, D.O., Sahu, D.R., (2007), Iterative construction of fixed points of nearly asymptotically nonexpansive mappings, J. Nonlinear Convex Anal., 8, pp.61-79.

[2] Altun, I., Olgun, M., (2020), An existence and uniqueness theorem for a fractional boundary value problem via new fixed point results on quasi-metric spaces, Commun. Nonlinear Sci., 91, 105462.

[3] Aoua, L B., Parvaneh, V., Oussaeif, T.E., Guran, L., Laid, G.H., Park, C., (2023), Common fixed point theorems in intuitionistic fuzzy metric spaces with an application for Volterra integral equations, Commun. Nonlinear Sci., Commun. Nonlinear Sci. Numer. Simul., 127, 107524.

[4] Baleanue, D., Rezapour, S., Mohammadi, H., (2012), Some existence results on nonlinear fractional differential equations, Philos. Trans. R. Soc. A, 371, 20120144.

[5] Berinde, V., (2007), Iterative Approximations of Fixed Points, Berlin Heidelberg: Springer Verlag, 326p.

[6] Debnath, P., Konwar, N., Radenovic, S., (2021), Metric Fixed Point Theory: Applications in Science, Engineering and Behavioural Sciences, Springer Singapore, 362p.

[7] Hacioglu, E., Gursoy, F., Maldar, S., Atalan, Y., Milovanovic, G.V., (2021), Iterative approximation of fixed points and applications to two-point second-order boundary value problems and to machine learning, Appl. Numer. Math., 167, pp.143-172.

[8] Hardy, G.E., Rogers, T.D., (1973), A generalized of a fixed point theorem of Reich, Canad.Math. Bull., 16, pp.201-206.

[9] Ishikawa, S., (1974), Fixed points by a new iteration method, Pro. Amer. Math. Soc., 44, pp.147-150.

[10] Karapinar, E., (2023), Recent advances on metric fixed point theory: a review, Appl. Comput. Math., 22(1), pp.3-30.

[11] Krasnoselskii, M.A., (1955), Two observations about the method of sucessive approximation, Usp. Math. Nauk., 10, pp.123- 127.

[12] Kumam, W., Khammahawong, K., Kumam, P., (2019), Error estimate of data dependence for discontinuous operators by new iteration process with convergence analysis, Numer. Funct. Anal. Optim., 40, pp.1644-1677.

[13] Malhotra1, A., Kumar, D., (2024), Existence and stability of solution for a nonlinear Volterra integral equation with binary relation via fixed point results, J. Comput. Appl. Math., 441, 115686.

[14] Maniu, G., (2020), On a three-step iteration process for Suzzuki mappings with qualitative study, Numer. Funct. Anal. Optim., 41, pp.929-949.

[15] Mann, W.R., (1953), Mean value methods in iteration, Pro. Amer. Math. Soc., 4, pp.506-510.

[16] Panja, S., Roy, K., Paunovi, M.V., Saha, M., Parvaneh, V., (2022), Fixed points of weakly K-nonexpansive mappings and a stability result for fixed point iterative process with an application, J. Inequal. Appl., 2022, 90.

[17] Picard, E., (1890), Memoire sur la theorie des equations aux derivees partielles et la methode des approximations successives, Journal de Mathematiques Pures et Appliquees, 6, pp.145-210, (in French).

[18] Shah, K., Abdeljawad, T., Ahmad, I., (2024), On non-linear fractional order coupled pantograph differential equations under nonlocal boundary conditions, TWMS J. Pure Appl. Math., 15(1), pp.65-78.

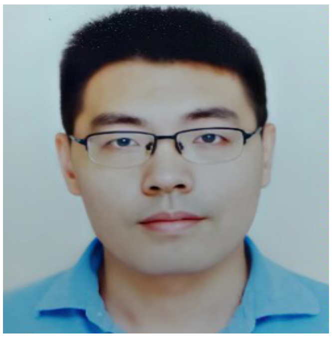
[19] Thakur, D., Thakur, B.S., Postolache, M., (2014), New iteration scheme for numerical reckoning fixed points of nonexpansive mappings, J. Inequal. Appl., 2014, 328.

[20] Wang, C., (2024), Solving nonlinear fractional differential equations by common fixed point results for a pair of (α, Θ)-type contractions in metric spaces, Demonstratio Math., 57, 20240081.

[21] Wang, C., Fan, H.L., (2022), A fixed point theorem for a pair of generalized nonexpansive mappings in uniformly convex metric spaces, J. Math. Study, 55, pp.432-444.

[22] Wang, C., Zhang, T.Z., (2016), Approximating common fixed point for a pair of generalized nonlinear mappings in convex metric space, J. Nonlinear Sci. Appl., 9, pp.1-7.

[23] Xu, B., Noor, M.A., (2002), Fixed point iterations for asymptotically nonexpansive mappings in Banach spaces, J. Math. Anal. Appl., 276, pp.444-453.



Chao Wang - received the B.S. in information and computing science and the M.S. degrees in applied mathematics from Nanchang University, Nanchang, Jiangxi, China, in 2004 and 2007, respectively, and Ph.D. degree in applied mathematics from Tongji University, Shanghai, China, in 2011. He is now an associate professor at the School of Mathematics and Statistics of Nanjing University of Information Science and Technology, Jiangsu, China. His research interests include nonlinear analysis, algorithms, fixed point theory and applications, and control theory.